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Homological Field Theory

Maria Kallimani
Joint work with Olaf Hohm,
Christoph Chiaffrino

Overview

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- BV algebras
- Relation to L_∞

4 Computing correlators

- Homotopy transfer
- Homological perturbation lemma

5 Outlook

Motivation

Main question

How to formulate the computation of **correlation functions** in a quantum field theory in an **algebraic** language?

Quantum Field Theory

Building blocks?

Algebraic description

$$Z = \int \mathscr{D}\phi^i \exp\left(-\frac{i}{\hbar}S[\phi^i]\right)$$

map?

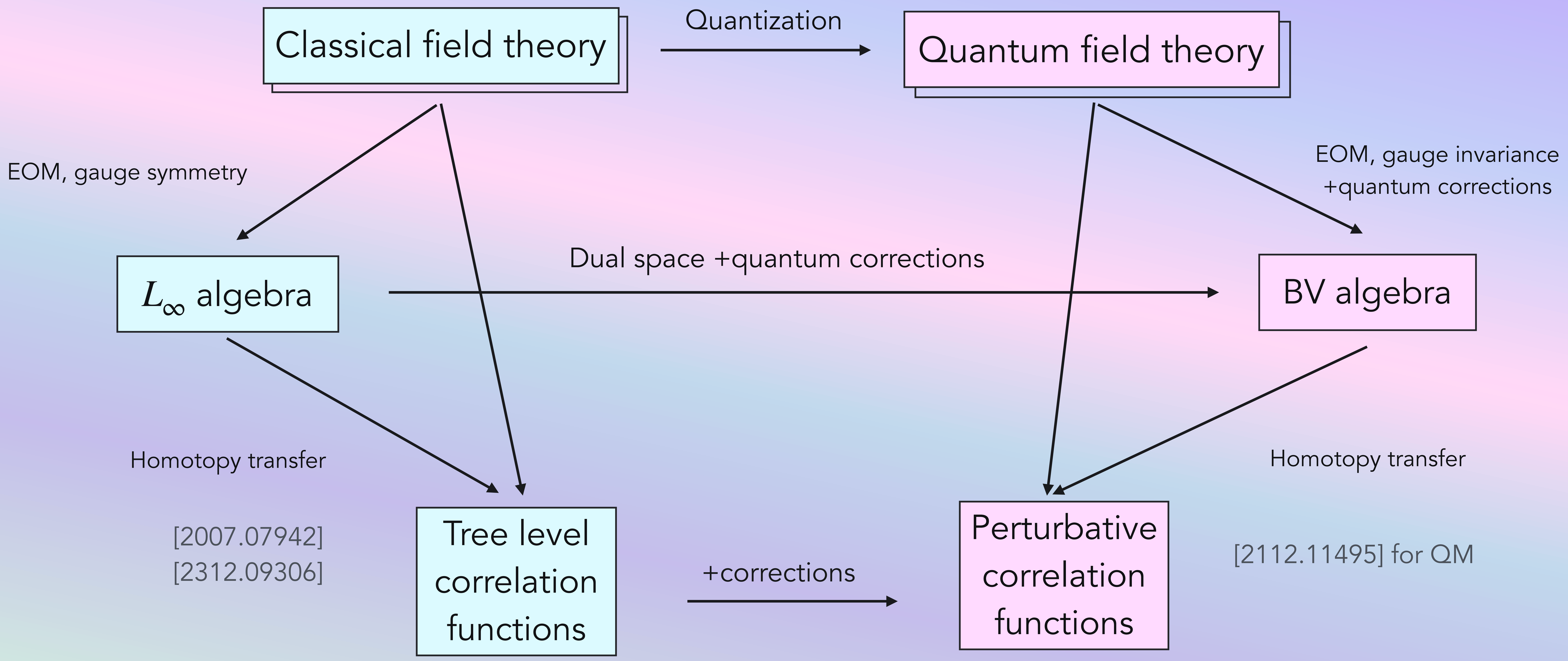
$$\int \mathscr{D}\phi^i \mathscr{O}(\phi^i) \exp\left(-\frac{i}{\hbar}S[\phi^i]\right) = \langle \mathscr{O} \rangle$$

Minimal space

Goal

Relate theories with equivalent physical content through maps between their algebraic descriptions

The idea



Classical field theory
Quantization
Quantum field theory
EOM, gauge symmetry
EOM, gauge invariance
+quantum corrections
Dual space +quantum corrections
L_∞ algebra
BV algebra
Homotopy transfer
Homotopy transfer
[2007.07942]
[2312.09306]
Tree level correlation functions
+corrections
Perturbative correlation functions
[2112.11495] for QM

Field theories as L_∞ algebras

Differential graded algebras

$\mathbb{Z}$-graded vector space

$$V = \bigoplus_{n\in\mathbb{Z}} V_n$$

$$v \in V_k \leftrightarrow |v| := k$$

+

Nilpotent differential

$$b_1 : V_n \to V_{n+1}$$

$$(b_1)^2 = 0$$

+

Product

Leibniz rule

$$[b_1, b_2] = 0$$

Chain complex

$$0 \to \dots \xrightarrow{b_1} V_{n-1} \xrightarrow{b_1} V_n \xrightarrow{b_1} V_{n+1} \xrightarrow{b_1} \dots \to 0$$

Cohomology

$$H^n(V) := \frac{\ker b_1|_{V_n}}{\mathrm{im} b_1|_{V_{n-1}}}$$

Differential graded Lie algebras

$\mathbb{Z}$-vector space

$$V = \oplus_{n\in\mathbb{Z}} V_n$$

$$v \in V_k \leftrightarrow |v| := k$$

+

Differential

$$b_1 : V_n \to V_{n+1}$$

$$(b_1)^2 = 0$$

+

Graded Lie bracket

$$b_2 : V \times V \to V, |b_2| = 1$$

$$b_2(v_1, v_2) = (-1)^{|v_1||v_2|} b_2(v_2, v_1)$$

Leibniz $[b_1, b_2] = 0$

Jacobi $b_2(b_2(v_1, v_2), v_3)$+(cyclic perm., graded) $= 0$

Chain complex

$$0 \to \dots \xrightarrow{b_1} V_{n-1} \xrightarrow{b_1} V_n \xrightarrow{b_1} V_{n+1} \xrightarrow{b_1} \dots \to 0$$

Cohomology

$$H^n(V) := \frac{\ker b_1|_{V_n}}{\mathrm{im} b_1|_{V_{n-1}}}$$

Example: G–valued forms

Setup 3 dimensional compact manifold M, Lie algebra $\mathfrak{g}$

Graded vector space	$V = \bigoplus_{m=0}^{3} \Omega^m(M) \otimes \mathfrak{g}[1]$
Differential	$b_1 = d \qquad d^2 = 0$
Lie bracket	$b_2(v_1, v_2) := [v_{1\mu}, v_{2\nu}](dx^\mu \wedge dx^\nu)$

Algebraic dictionary

Setup Chern-Simons theory

$$S[A] = \int_M Tr\left(\frac{1}{2}A \wedge dA + \frac{1}{3!}A \wedge [A, A]\right)$$

$$A \in \Omega(M, \mathfrak{g})$$

$$\downarrow$$

$dA + \frac{1}{2}[A, A] = 0$ +expand around background

Algebraic dictionary

Setup — Chern-Simons theory

$$0 \to V_{-1} \xrightarrow{b_1 = d} V_0 \xrightarrow{d} V_1 \xrightarrow{d} V_2 \to 0$$

Gauge parameters	Fields	Equations of Motion	Noether identities
$\lambda \in V_{-1} = \Omega^0(M) \otimes \mathfrak{g}[1]$	$A \in V_0 = \Omega^1(M) \otimes \mathfrak{g}[1]$	$\mathscr{E} \in V_1 = \Omega^2(M) \otimes \mathfrak{g}[1]$	$\mathscr{N} \in V_2 = \Omega^3(M) \otimes \mathfrak{g}[1]$

⊕ Cyclic inner product ⟶ Action

$$S[A] = \int_M Tr\left(\frac{1}{2} A \wedge dA + \frac{1}{3!} A \wedge [A, A]\right) \longrightarrow \omega(v_1, v_2) = \int_M Tr(v_1 \wedge v_2)$$

L_∞ algebras

Graded vector space

$$V = \oplus_{n\in\mathbb{Z}} V_n$$

$$v \in V_k \leftrightarrow |v| := k$$

Differential

$$b_1 : V_n \to V_{n+1}$$

$$(b_1)^2 = 0$$

Graded Lie bracket

$$b_2 : V \times V \to V, |b_2| = 1$$

$$b_2(v_1, v_2) = (-1)^{|v_1||v_2|} b_2(v_2, v_1)$$

Leibniz $b_1(b_2(v_1, v_2)) + b_2(b_1(v_1), v_2) + (-1)^{v_1 v_2}(1 \leftrightarrow 2) = 0$

Jacobi Up to homotopy!

$b_2(b_2(v_1, v_2), v_3)$+(cyclic perm., graded)

$+b_1 b_3(v_1, v_2, v_3) + b_3(b_1(v_1), v_2, v_3)$+(c.perm., graded)

$= 0$

Infinitely more brackets

$$b_n : V \times \ldots \times V \to V, |b_n| = 1$$

$$b_n(v_1, \ldots, v_n) = \mathrm{sgn}(\sigma) b_n(v_{\sigma(1)}, \ldots, v_{\sigma(n)})$$

$$D := \sum_n b_n$$

$$D^2 = 0$$

Algebraic dictionary

Setup

Perturbative field theory with gauge algebra $\mathfrak{g}$

(+Theory with interactions of order > 3, Lie algebra that doesn't close…)

$$\dots \to V_{-1} \xrightarrow{b_1} V_0 \xrightarrow{b_1} V_1 \xrightarrow{b_1} V_2 \to \dots$$

Gauge-for-gauge

Gauge parameters

$\lambda \in V_{-1}$

Fields

$A \in V_0$

Equations of Motion

$\mathcal{E} \in V_1$

Noether identities

$\mathcal{N} \in V_2$

Noether-for-Noether

How to quantise?

The path integral

$$Z = \int \mathscr{D}\phi^i \exp\left(-\frac{i}{\hbar}S[\phi^i]\right)$$

Field configurations

$$\langle F \rangle = \frac{1}{Z}\int \mathscr{D}\phi^i \exp\left(-\frac{i}{\hbar}S[\phi^i]\right) F$$

How to compute an expectation value?

People braver than me

Actually do the integral

In finite dimensions and with nice computers

Me: I only know how to do Gaussian integrals

Never do the integral

Use as a tool to derive equations $\left\langle \frac{\delta F}{\delta \phi^i} \right\rangle = i\hbar \left\langle \frac{\delta S}{\delta \phi^i} F \right\rangle$

People braver than me

Never do the integral

Rephrase the problem

Measure ρ ⟷ Divergence Δ

Δ-exact: no contribution (Stokes)

Free to gauge fix: $\Delta(\rho F) = 0$

BV algebras

Graded algebra

$(X = \bigoplus_{n\in\mathbb{Z}} X_n, \cdot\,)$

+

Second order differential: BV Laplacian

$\Delta : X \to X$

$\Delta^2 = 0$

$$\Delta(f\cdot g\cdot r) = -(\Delta f)\cdot g\cdot r - (-1)^f f\cdot(\Delta g)\cdot r + (-1)^{f+g} f\cdot g\cdot \Delta r$$
$$+\Delta(f\cdot g)r + (-1)^f f\cdot \Delta(g\cdot r) + (-1)^{(f+1)g} g\cdot \Delta(f\cdot r)$$

This hides more information:

Antibracket Failure of Δ to be first order

$\{\,\cdot\,,\cdot\,\} : X\times X \to X$

$\{f,g\} = (-1)^f\Delta(fg) - (-1)^f(\Delta f)g - f\Delta g$

Differential $S \in X_0$ $\quad \delta := \{S,\cdot\,\} - i\hbar\Delta =: Q - i\hbar\Delta$

QME $\quad \boxed{\frac{1}{2}\{S,F\} + i\hbar\Delta F = 0} \leftarrow \boxed{\Delta(F\exp(-i\hbar S)) = 0}$

From L_∞ to BV

""""dual"""" descriptions (locally, classically)

L_∞	BV
Coalgebra	Algebra
Coderivation $D = \sum_{n=0} b_n$, $D^2 = 0$	Derivation $Q = \{S, \cdot\ \}$, $Q^2 = 0$ (classically)
Space of fields V	Space of functionals on fields $\mathscr{F}(V) = Sym(C_c^\infty(V))$
Cyclic structure	Symplectic structure (secretly gives Δ)

Operators

Example: Chern-Simons theory

Start from $(V = \oplus_{n=-1}^{2} \Omega^{n+1}(M) \otimes \mathfrak{g}, b_1 = d \otimes 1, b_2 = \pm(\wedge) \otimes [,])$

with cyclic structure $\omega(v_1, v_2) = \int_M Tr(v_1 \wedge v_2)$

Functionals of fields $F[v] := \int_M \langle f, v \rangle$ where $v \in V$, $f \in \Omega_c(M) \otimes \mathfrak{g}^*$

Product $(F_1 \wedge F_2)[v_1, v_2] = F_1[v_1]F_2[v_2] + (-1)^{f_1 f_2} F_1[v_2]F_2[v_1]$

BV laplacian $\Delta(F_1 \wedge F_2) := \int_M Tr(f_1 \wedge f_2), \Delta(F[v]) = 0$

Differential $(Q_{free}F)[v] := F[b_1(v)] = F[dv]$

Observables

Physical information of a theory

Classical Theory

Solutions of equations of motion $\in H^0_D(V)$

$$[\phi^i] : \phi^i \sim \phi^i + \delta_\lambda \phi^i, \; \sum_{n=0} \frac{1}{n!} b_n(\phi^i, \ldots, \phi^i) = 0$$

Quantum Theory

Correlation functions of gauge invariant functionals $\in H^0_\delta(\mathscr{F}(V))$

$$\Delta\left(F \exp\left(\frac{i}{\hbar}S\right)\right) = 0, \; [F[\phi^i]] \equiv \langle F[\phi^i]\rangle$$

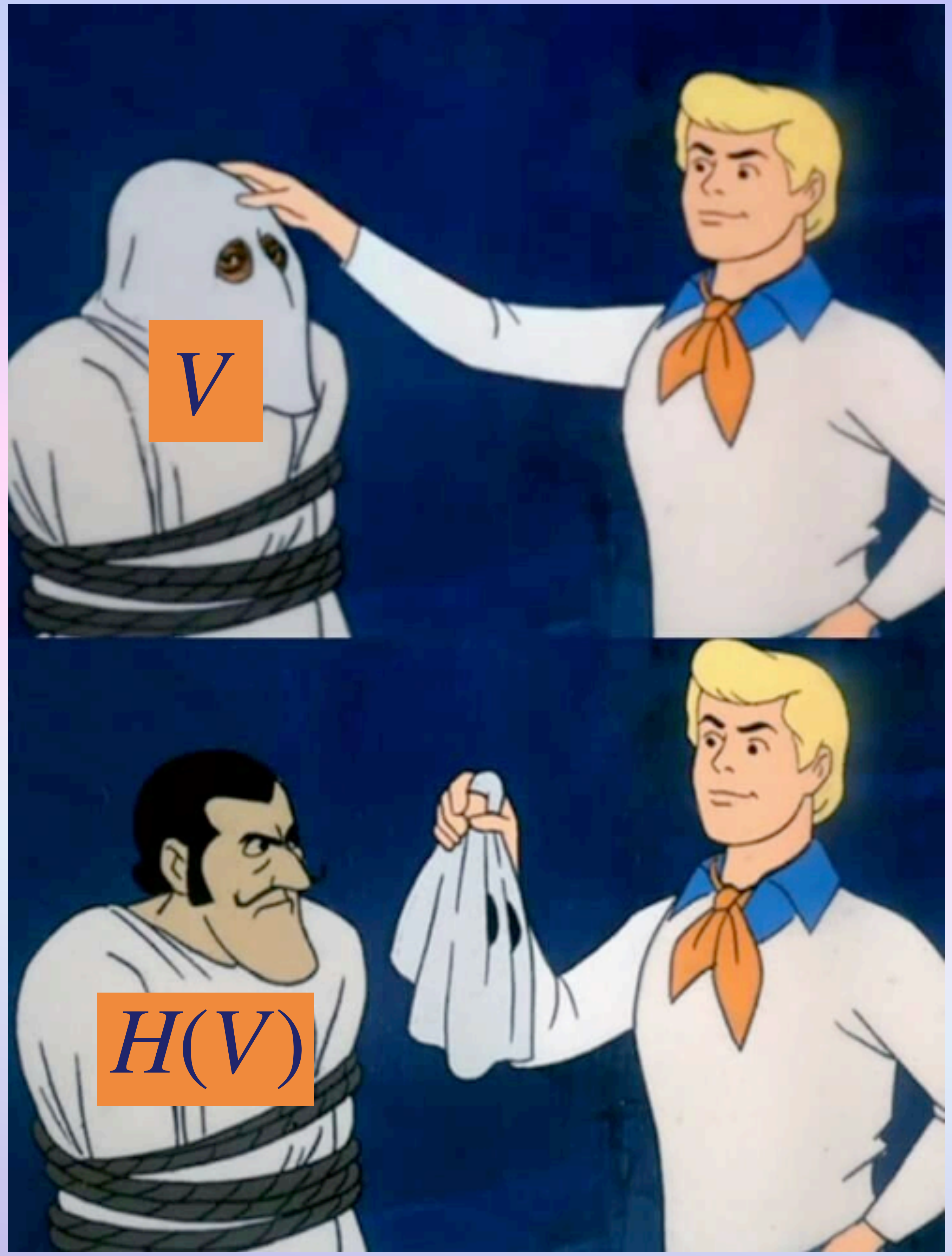
Extracting physical information $\longleftrightarrow$ Going to cohomology

Homotopy transfer

Map between original theory and one with *equivalent physical information*

$$
\begin{tikzcd}
V: 0 \arrow[r] & \ldots \arrow[r, "b_1"] & V_{n-1} \arrow[r, "b_1"] \arrow[l, bend right, "h"'] \arrow[d, shift left, "p"] & V_n \arrow[r, "b_1"] \arrow[l, bend right, "h"'] \arrow[d, shift left, "p"] & V_{n+1} \arrow[r, "b_1"] \arrow[l, bend right, "h"'] \arrow[d, shift left, "p"] & \ldots \arrow[r] & 0 \\
\bar{V}: 0 \arrow[r] & \ldots \arrow[r, "\bar{b}_1"] & \bar{V}_{n-1} \arrow[r, "\bar{b}_1"] \arrow[u, shift left, "i"] & \bar{V}_n \arrow[r, "\bar{b}_1"] \arrow[u, shift left, "i"] & \bar{V}_{n+1} \arrow[r, "\bar{b}_1"] \arrow[u, shift left, "i"] & \ldots \arrow[r] & 0
\end{tikzcd}
$$

Special case: map to minimal complex, $\bar{b}_1 = 0$ $\rightarrow H_{b_1}(V)$



Homotopy transfer

Map between original theory and one with *equivalent physical information*

$$\begin{tikzcd}
V: 0 \arrow[r] & \dots \arrow[r, "b_1"] & V_{n-1} \arrow[r, "b_1"] \arrow[d, shift right, "i"'] & V_n \arrow[r, "b_1"] \arrow[l, bend right, "h"'] \arrow[d, shift right, "i"'] & V_{n+1} \arrow[r, "b_1"] \arrow[l, bend right, "h"'] \arrow[d, shift right, "i"'] & \dots \arrow[r] \arrow[l, bend right, "h"'] & 0 \\
\bar{V}: 0 \arrow[r] & \dots \arrow[r, "\bar{b}_1"] & \bar{V}_{n-1} \arrow[r, "\bar{b}_1"] \arrow[u, shift right, "p"'] & \bar{V}_n \arrow[r, "\bar{b}_1"] \arrow[u, shift right, "p"'] & \bar{V}_{n+1} \arrow[r, "\bar{b}_1"] \arrow[u, shift right, "p"'] & \dots \arrow[r] & 0
\end{tikzcd}$$

Quasi-isomorphism:

$$\boxed{hb_1 + b_1 h = 1_V - ip}$$

Special case: map to minimal complex, $\bar{b}_1 = 0 \ \rightarrow H_{b_1}(V)$

Lift the homotopy to $\mathscr{F}(V)$: $\quad (HF)[v] := F[h(v)]$

Retract of $\mathscr{F}(V)$ to $H_{Q_{free}}(F(V))$:

$$((\bar{Q}_{free}H + HQ_{free})F)[v] := F[(1 - ip)(v)] \equiv ((1 - IP)F)[v]$$

Homological perturbation lemma

Minimal complex deformed by non linear terms in differential

$$\delta \equiv Q_{free} + \underbrace{Q_{int}}_{\text{Interactions}} + \underbrace{i\hbar\Delta}_{\text{Quantization}}$$

Perturbative case:

$$\boxed{\underbrace{[F]}_{\color{magenta}{\langle F\rangle}} = P \sum_{n=0} \left(- (\underbrace{Q_{int}}_{\text{Source derivatives}} + \underbrace{i\hbar\Delta}_{\text{Wick contractions}})H\right)^n F} \longrightarrow \langle F\rangle \sim \frac{\delta}{\delta J}\int \mathscr{D}\phi^i \exp\left(\frac{i}{\hbar}S[\phi^i] + \langle J, F\rangle\right)$$

Outlook

Factorisation algebras?

Retract to $H_{Q_{free}}(\mathscr{F}(V))$
not algebra morphism
(Δ: second order)

But *is* for functionals with disjoint support

Boundaries?

$$Z = \int \mathscr{D}\phi^i \exp\left(-\frac{i}{\hbar}S[\phi^i]\right)$$

field+boundary conditions

Changes in cyclic structure/Δ to account for boundary terms

Large N?

Capture 1/N expansion with appropriate differential

$$N \to \infty : H_?(V?)$$

Thank you for your attention!

Bibliography

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- Roberto Bonezzi, Christoph Chiaffrino, Felipe Diaz-Jaramillo and Olaf Hohm, *"Tree-level scattering amplitudes via Homotopy Transfer"*, arXiv: 2312.09306

- Christoph Chriaffrino, Olaf Hohm, Alison F. Pinto, *"Homological Quantum Mechanics"* arXiv: 2112.11495

dgA: de Rham complex

Setup 3 dimensional compact Euclidean manifold M

Graded vector space $V = \oplus_{m=0}^{3} \Omega^m(M)$

Associative product $v_1 \wedge v_2 = (-1)^{v_1 v_2} v_2 \wedge v_1$

Differential $d := \partial_\mu dx^\mu \wedge \qquad d^2 = 0$

$d(v_1 \wedge v_2) - dv_1 \wedge v_2 - (-1)^{v_1} v_1 \wedge dv_2 = 0$

Cohomology $H^m(M) \simeq Harm^m(M)$

Algebraic dictionary

Setup

Chern-Simons theory

Gauge transformations

$\lambda \in V_{-1}$

$A \in V_0$

$$\delta_\lambda A = d\lambda + [A, \lambda] \equiv b_1(\lambda) + b_2(\lambda, A)$$

$$\delta_{-[\lambda_1, \lambda_2]} = [\delta_{\lambda_1}, \delta_{\lambda_2}] \equiv b_2(\lambda_1, \lambda_2)$$

Equations of motion

$A \in V_0$

$$dA + \frac{1}{2}[A, A] \equiv b_1(A) + \frac{1}{2} b_2(A, A) \overset{!}{=} 0$$

Algebraic dictionary

Setup — Perturbative field theory with gauge algebra $\mathfrak{g}$

Gauge transformations — $\lambda \in V_{-1}$, $A \in V_0$

$$\delta_\lambda A = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} b_n(\lambda, A, \ldots, A)$$

Equations of motion — $A \in V_0$

$$\sum_{n=1}^{\infty} \frac{1}{n!} b_n(A, \ldots, A) \stackrel{!}{=} 0$$

Cyclic L_∞ algebras

L_∞ algebra ⊕ Cyclic inner product

$$\omega : V \times V \to \mathbb{R}$$

$$\omega(v_1, v_2) = (-1)^{v_1 v_2 + 1}$$

$$\omega(b_n(v_1, \ldots, v_n), v_{n+1}) = (-1)^{v_{n+1} \sum_{k=1}^{n} v_k} \omega(b_n(v_{n+1}, v_1, \ldots, v_{n-1}), v_n)$$

Algebraic dictionary

Setup

Perturbative *Lagrangian* field theories

Action

$A \in V_0$

$$S[A] = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \omega(A, b_n(A, \ldots, A))$$

Setup

Chern-Simons theory

$$S[A] = \int_M Tr\left(\frac{1}{2} A \wedge dA + \frac{1}{3!} A \wedge [A, A]\right) \longrightarrow \omega(v_1, v_2) = \int_M Tr(v_1 \wedge v_2)$$

From measures to divergences

Finite dimensions: $\{\phi^1, \dots, \phi^n\} \longrightarrow$ Volume form: $\rho d\phi^1 \wedge \dots \wedge d\phi^n$

Divergence: $div_\rho(V^i) = \frac{1}{\rho}\frac{\partial}{\partial \phi^i}(\rho V^i)$

Stoke's theorem: $\int d^n\phi \rho (div_\rho F) = 0$ $\quad F = \tilde{F} + div_\rho(F_e) \longrightarrow I(F) = \int d^n\phi\rho F$ depends only on $\tilde{F}$

For the path integral: $\rho = \exp\left(\frac{i}{\hbar}S\right) \qquad \langle F \rangle = \frac{I(F)}{I(1)}$

$$\text{Space of F} \xrightarrow[\langle \cdot \rangle]{} \text{Space of F} \big/ div_\rho\text{-exact terms}$$

Typical BV

V: space of fields (+ghosts), $dimV = n$ $\xrightarrow[\text{+conjugate}]{}$ $V \oplus \Pi V^*[1]$: +anti fields (+anti ghosts)

BV manifold

$\oplus$

Second order differential

Symplectic structure $\omega = d\phi^i \wedge d\phi_i^+$ $\longleftrightarrow$ Laplacian: $\Delta = \dfrac{\delta^2}{\delta\phi^+\delta\phi}$, $\Delta^2 = 0$

Lagrangian submanifold: $\mathscr{L} \subset V \oplus \Pi V^*[1]$, $\omega|_{\mathscr{L}} = 0$, $dim\mathscr{L} = n$

$$\int_{\mathscr{L}} F$$

$\Delta F = 0$: depends only on homology of $\mathscr{L}$

$F = \Delta G$: is always 0

- For a path integral to be independent of gauge choice, we must have $\Delta\left(\exp(\frac{i}{\hbar}S)\right) = 0$
- Δ-exact pieces do not contribute: we can change S by canonical transformations.